

Ramsey numbers for multiple copies of hypergraphs

G.R. Omid^{a,c,1} G. Raeisi^{b,c,2}

^aDepartment of Mathematical Sciences, Isfahan University of Technology,
Isfahan, 84156-83111, Iran

^bDepartment of Mathematics, Shahrekord University,
Shahrekord, P.O. Box 115, Iran

^cSchool of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran

romidi@cc.iut.ac.ir g.raeisi@sci.sku.ac.ir, g.raeisi@math.iut.ac.ir

Abstract

In this paper, for sufficiently large n we determine the Ramsey number $R(\mathcal{G}, n\mathcal{H})$ where \mathcal{G} is a k -uniform hypergraph with the maximum independent set that intersects each of the edges in $k - 1$ vertices and \mathcal{H} is a k -uniform hypergraph with a vertex so that the hypergraph induced by the edges containing this vertex is a star. There are several examples for such \mathcal{G} and \mathcal{H} , among them are any disjoint union of k -uniform hypergraphs involving loose paths, loose cycles, tight paths, tight cycles with a multiple of k edges, stars, Kneser hypergraphs and complete k -uniform k -partite hypergraphs for \mathcal{G} and linear hypergraphs for \mathcal{H} . As an application, $R(m\mathcal{G}, n\mathcal{H})$ is determined where m or n is large and \mathcal{G} and \mathcal{H} are either loose paths, loose cycles, tight paths, or stars. Also, $R(\mathcal{G}, n\mathcal{H})$ is determined when \mathcal{G} is a bipartite graph with a matching saturating one of its color classes and \mathcal{H} is an arbitrary graph for sufficiently large n . Moreover, some bounds are given for $R(m\mathcal{G}, n\mathcal{H})$ which allow us to determine this Ramsey number when $m \geq n$ and \mathcal{G} and \mathcal{H} ($|V(\mathcal{G})| \geq |V(\mathcal{H})|$) are 3-uniform loose paths or cycles, k -uniform loose paths or cycles with at most 4 edges and k -uniform stars with 3 edges.

Keywords: Ramsey number, Hypergraph, Loose cycle, Loose path.

AMS Subject Classification: 05C15, 05C55, 05C65.

¹This research is partially carried out in the IPM-Isfahan Branch and in part supported by a grant from IPM (No. 91050416).

²This research was in part supported by a grant from IPM (No.91050018).

1 Introduction

A k -uniform hypergraph \mathcal{H} is a pair (V, E) where V is a set of vertices and E is a set of k -subsets of V (the edges of \mathcal{H}). A hypergraph \mathcal{H} is *linear* if the intersection of every two edges of \mathcal{H} has at most one element. As usual, the complete k -uniform hypergraph on p vertices is denoted by \mathcal{K}_p^k and for a given hypergraph \mathcal{H} , $n\mathcal{H}$ is used to denote the n disjoint copies of \mathcal{H} . For a hypergraph \mathcal{H} , a set $S \subseteq V(\mathcal{H})$ is called *independent* if there is no edge of \mathcal{H} contained in S . The *independence number* of \mathcal{H} , denoted by $\alpha(\mathcal{H})$, is the size of the greatest independent set in \mathcal{H} .

There are several natural definitions for a cycle and a path in uniform hypergraphs. Those we focus on here are called *loose* and *tight*. By a k -uniform *loose cycle* \mathcal{C}_n^k (resp. *tight cycle* $\hat{\mathcal{C}}_n^k$), we mean the hypergraph with vertex set $\{v_1, v_2, \dots, v_{n(k-1)}\}$ (resp. $\{v_1, v_2, \dots, v_n\}$) and with the set of n edges $e_i = \{v_1, v_2, \dots, v_k\} + i(k-1)$, $i = 0, 1, \dots, n-1$ (resp. $e_i = \{v_1, v_2, \dots, v_k\} + i$, $i = 0, 1, \dots, n-1$), where we use mod $n(k-1)$ (resp. mod n) arithmetic and adding a number t to a set $S = \{v_1, v_2, \dots, v_k\}$ means a shift, i.e., the set obtained by adding t to subscripts of each element of S . Similarly, a k -uniform *loose path* \mathcal{P}_n^k (resp. *tight path* $\hat{\mathcal{P}}_n^k$), is the hypergraph with vertex set $\{v_1, v_2, \dots, v_{n(k-1)+1}\}$ (resp. $\{v_1, v_2, \dots, v_{n+k-1}\}$) and with the set of n edges $e_i = \{v_1, v_2, \dots, v_k\} + i(k-1)$, $i = 0, 1, \dots, n-1$ (resp. $e_i = \{v_1, v_2, \dots, v_k\} + i$, $i = 0, 1, \dots, n-1$). Also, by a *star* \mathcal{S}_n^k we mean the k -uniform hypergraph with vertex set $\{v, v_1, v_2, \dots, v_{n(k-1)}\}$ and with the set of n edges $e_i = \{v\} \cup (\{v_1, v_2, \dots, v_{k-1}\} + i(k-1))$, $i = 0, 1, \dots, n-1$. For $k = 2$ we get the usual definitions of a cycle C_n , a path P_n and a star $K_{1,n}$ with n edges.

For any given k -uniform hypergraphs \mathcal{G} and \mathcal{H} , the *Ramsey number* $R(\mathcal{G}, \mathcal{H})$ is the smallest positive integer N such that in every red-blue coloring of the edges of the complete k -uniform hypergraph on N vertices there is a monochromatic copy of \mathcal{G} in color red or a monochromatic copy of \mathcal{H} in color blue. The existence of such a positive integer is guaranteed by Ramsey's classical result in [14]. The Ramsey number of graphs involving cycles and paths are completely known (See [4, 5, 6, 15]). For the Ramsey number of tight paths and cycles the results in [10] give the asymptotic behaviors of $R(\hat{\mathcal{C}}_n^3, \hat{\mathcal{C}}_n^3)$ and $R(\hat{\mathcal{P}}_n^3, \hat{\mathcal{P}}_n^3)$. The Ramsey problem for loose paths and cycles were investigated by several authors. It was proved in [9] that $R(\mathcal{C}_n^3, \mathcal{C}_n^3)$ is asymptotically equal to $\frac{5n}{2}$. Subsequently, Gyárfás *et. al.* in [8] extended this result to k -uniform loose cycles and proved that $R(\mathcal{C}_n^k, \mathcal{C}_n^k)$ is asymptotically equal to $\frac{1}{2}(2k-1)n$. The proofs of all of these results are based on the method of the Regularity Lemma. In [7], the authors determined the exact values of the Ramsey numbers of k -uniform loose triangles and quadrangles.

Theorem 1.1 ([7]) *For every $k \geq 3$, $R(\mathcal{P}_3^k, \mathcal{P}_3^k) - 1 = R(\mathcal{C}_3^k, \mathcal{C}_3^k) = 3k - 2$ and $R(\mathcal{P}_4^k, \mathcal{P}_4^k) - 1 = R(\mathcal{C}_4^k, \mathcal{C}_4^k) = 4k - 3$.*

Also the Ramsey number of 3-uniform loose paths is determined when one of the paths is significantly larger than the other. In the other words, it is proved in [11] that if $r \geq \lfloor \frac{5s}{4} \rfloor$, then $R(\mathcal{P}_r^3, \mathcal{P}_s^3) = 2r + \lfloor \frac{s+1}{2} \rfloor$. Recently in [12], the exact values of the Ramsey numbers of 3-uniform hypergraphs involving loose cycles and paths have been determined as follows.

Theorem 1.2 ([12]) *For every $r \geq s$, $R(\mathcal{P}_r^3, \mathcal{P}_s^3) = R(\mathcal{C}_r^3, \mathcal{C}_s^3) + 1 = R(\mathcal{P}_r^3, \mathcal{C}_s^3) = 2r + \lfloor \frac{s+1}{2} \rfloor$. Moreover, $R(\mathcal{C}_r^3, \mathcal{P}_s^3) = 2r + \lfloor \frac{s-1}{2} \rfloor$ if $r > s$.*

The problem of determining the Ramsey numbers for multiple copies of graphs was first studied by Burr *et al.* in [1] where the authors found the Ramsey numbers for multiple copies of triangles and stars. More generally, Burr *et al.* gave the following result on the Ramsey number of connected graphs.

Theorem 1.3 ([1]) *Let G and H be connected graphs. Then there is a constant c , depending only on G and H , such that for sufficiently large n*

$$R(nG, nH) = (|V(G)| + |V(H)| - \min\{\alpha(G), \alpha(H)\})n + c.$$

In [2] and [3], Burr developed much more powerful techniques to investigate the behavior of $R(nG, mH)$ when either m or n is large. In particular, Burr proved for fixed $r, s \geq 4$ and sufficiently large m or n , that

$$R(mC_r, nC_s) = m|V(C_r)| + n|V(C_s)| - \min\{m\alpha(C_r), n\alpha(C_s)\} - 1. \quad (1)$$

It is non-trivial to give a new version of Theorem 1.3 for uniform hypergraphs, here we do this by assuming some additional conditions. We give the exact value of $R(\mathcal{G}, n\mathcal{H})$ where \mathcal{G} is a k -uniform hypergraph with the maximum independent set that intersects each of the edges in $k-1$ vertices, \mathcal{H} has a vertex so that the hypergraph induced by the edges containing this vertex is a star and n is sufficiently large. (See Theorem 2.7.) Such evaluations are often possible in practice, as shown by several examples; for instance, $R(m\mathcal{C}_r^k, n\mathcal{C}_s^k)$, $R(m\mathcal{C}_r^k, n\mathcal{P}_s^k)$, $R(m\mathcal{P}_r^k, n\mathcal{P}_s^k)$, $R(m\hat{\mathcal{P}}_r^k, n\hat{\mathcal{P}}_s^k)$ and $R(m\mathcal{S}_r^k, n\mathcal{S}_s^k)$ are determined where m or n is large and $k \geq 3$. (See Theorem 2.9.) All these would satisfy a formula similar to (1) with obvious substitutions. As another example, $R(\mathcal{G}, n\mathcal{H})$ is determined for a bipartite graph \mathcal{G} with a matching saturating one of its color classes, an arbitrary graph \mathcal{H} and sufficiently large n . Moreover, in Section 3, we give the exact values of Ramsey numbers for various cases; for instance $R(m\mathcal{G}, n\mathcal{H})$ is determined in the case when $m \geq n$ and \mathcal{G} and \mathcal{H} ($|V(\mathcal{G})| \geq |V(\mathcal{H})|$) are 3-uniform loose paths or cycles, k -uniform loose paths or cycles with at most 4 edges and k -uniform stars with 3 edges.

2 $R(\mathcal{G}, n\mathcal{H})$ for large n

We begin with some definitions and notations. Let \mathcal{H} be a hypergraph and $S \subseteq V(\mathcal{H})$ (resp. $S \subseteq E(\mathcal{H})$). By the *induced hypergraph on S* , denoted by $\langle S \rangle$, we mean the hypergraph with vertex set S and those edges of \mathcal{H} which are contained in S (resp. with vertex set $\bigcup_{e \in S} e$ and the edge set S). In the sequel, for a 2-edge coloring of a uniform hypergraph \mathcal{H} , say red and blue, we denote by \mathcal{H}_{red} and \mathcal{H}_{blue} the induced hypergraph on edges of color red and blue, respectively. A *matching* in a hypergraph \mathcal{H} is a set of mutually disjoint edges and the *matching number*, $\nu(\mathcal{H})$, is defined as the size of the largest matching. A *covering* in \mathcal{H} is a set $S \subseteq V(\mathcal{H})$ such that any edge of \mathcal{H} intersects S . The *covering number* of \mathcal{H} , $\tau(\mathcal{H})$, is defined as the size of the smallest covering in \mathcal{H} .

Another useful important variant in this paper is the *strong independent set*. A *strong independent set* of a k -uniform hypergraph \mathcal{H} is an independent subset S of vertices such that each edge of \mathcal{H} has exactly one vertex outside S . We denote by \mathcal{F}_k , the set of all k -uniform hypergraphs which have a strong independent set. If $\mathcal{H} \in \mathcal{F}_k$, the cardinality of the largest

strong independent set of \mathcal{H} is called the *strong independence number* and is denoted by $\alpha^*(\mathcal{H})$. One can easily see that \mathcal{F}_2 is the set of all bipartite graphs and for each $\mathcal{G} \in \mathcal{F}_2$, $\alpha^*(\mathcal{G})$ is the size of the largest color class of \mathcal{G} in all proper 2-colorings of $V(\mathcal{G})$. Clearly $\alpha^*(\mathcal{H}) \leq \alpha(\mathcal{H})$ for each $\mathcal{H} \in \mathcal{F}_k$. A hypergraph $\mathcal{H} \in \mathcal{F}_k$ is called *good* if $\alpha^*(\mathcal{H}) = \alpha(\mathcal{H})$. We denote by \mathcal{G}_k , the set of all good k -uniform hypergraphs. Clearly \mathcal{G}_k is closed under disjoint union, that is the disjoint union of every two hypergraphs in \mathcal{G}_k is in \mathcal{G}_k . The following is a characterization of good uniform hypergraphs.

Theorem 2.1 *Let \mathcal{H} be a uniform hypergraph. Then $\mathcal{H} \in \mathcal{G}_k$ if and only if $V(\mathcal{H})$ can be partitioned into two subsets V_1 and V_2 so that each edge has one vertex in V_1 and $k-1$ vertices in V_2 and for each $S \subseteq V_1$,*

$$\tau(\mathcal{H}_S) \geq |S|,$$

where \mathcal{H}_S is the $k-1$ -uniform hypergraph obtained from \mathcal{H} by deleting the vertices of S from those edges of \mathcal{H} which have nonempty intersection with S .

Proof. First, let $\mathcal{H} \in \mathcal{G}_k$. By the definition, the vertices of \mathcal{H} can be partitioned into two subsets V_1 and V_2 so that each edge has one vertex in V_1 and $k-1$ vertices in V_2 and $|V_2| = \alpha^*(\mathcal{H}) = \alpha(\mathcal{H})$. Now, on contrary suppose that there is a set $S \subseteq V_1$ with $|\tau(\mathcal{H}_S)| < |S|$ and assume that $S' \subseteq V(\mathcal{H}_S)$ is the minimum covering of \mathcal{H}_S . Clearly $V_2 \cup S - S'$ is an independent set of \mathcal{H} with more than $|V_2| = \alpha(\mathcal{H})$ vertices, a contradiction.

To prove the converse assume that $V(\mathcal{H})$ can be partitioned into two subsets V_1 and V_2 so that each edge has one vertex in V_1 and $k-1$ vertices in V_2 and for each $S \subseteq V_1$, we have $\tau(\mathcal{H}_S) \geq |S|$. Suppose that $U \subseteq V(\mathcal{H})$ is the maximum independent set, $S = V_1 \cap U$ and $S' = V(\mathcal{H}_S) - U$ is the covering of \mathcal{H}_S . Clearly $V_2 \cup S - S'$ is an independent set of \mathcal{H} and $U \subseteq V_2 \cup S - S'$ and so

$$|U| = |V_2 \cup S - S'| = |V_2| + |S| - |S'| \leq |V_2|.$$

Since V_2 is an independent set we conclude that $|V_2| = \alpha(\mathcal{H})$ and since $|V_2| \leq \alpha^*(\mathcal{H}) \leq \alpha(\mathcal{H})$ we have $\mathcal{H} \in \mathcal{G}_k$. ■

A hypergraph is *r-regular* if each of its vertices lies on the r edges. The following corollary shows that each regular element in \mathcal{F}_k is good.

Corollary 2.2 *Let \mathcal{H} be r -regular and $\mathcal{H} \in \mathcal{F}_k$. Then $\mathcal{H} \in \mathcal{G}_k$.*

Proof. Let Y be the maximum strong independence set of \mathcal{H} and $X = V(\mathcal{H}) - Y$. Hence each edge has one vertex in X and $k-1$ vertices in Y . Now, let $S \subseteq X$. Clearly \mathcal{H}_S has $r|S|$ edges and since \mathcal{H} is r -regular the number of edges of \mathcal{H}_S containing a vertex in $V(\mathcal{H}_S)$ is at most r . Hence $r\tau(\mathcal{H}_S) \geq r|S|$ and so the proof is complete by Theorem 2.1. ■

For $k=2$ the condition $|\tau(\mathcal{H}_S)| \geq |S|$ in Theorem 2.1 is equivalent to the Hall condition for the existence a matching saturating V_1 in a bipartite graph \mathcal{H} . So we have the following result.

Corollary 2.3 *$\mathcal{H} \in \mathcal{G}_2$ if and only if \mathcal{H} is a bipartite graph with a matching saturating one of its color classes.*

In the following we give some examples of good hypergraphs.

Remark 2.4

Let \mathcal{C}_n^k be the k -uniform loose cycle with vertex set $\{v_1, v_2, \dots, v_{n(k-1)}\}$ and the set of n edges $e_i = \{v_1, v_2, \dots, v_k\} + i(k-1)$ where $i = 0, 1, \dots, n-1$. It is clear to see that if n is even, then the set

$$S = \{v_1, v_{2k-1}, v_{4k-3}, \dots, v_{(n-2)(k-1)+1}\},$$

is a covering and for odd n the set

$$S = \{v_1, v_{2k-1}, v_{4k-3}, \dots, v_{(n-3)(k-1)+1}, v_{(n-1)(k-1)}\},$$

is a covering for \mathcal{C}_n^k . In both cases, $T = V(\mathcal{C}_n^k) \setminus S$ with $|T| = (k-1)n - \lfloor \frac{n+1}{2} \rfloor$ is a maximum strong independent set (also a maximum independent set). Therefore $\alpha^*(\mathcal{C}_n^k) = \alpha(\mathcal{C}_n^k) = (k-1)n - \lfloor \frac{n+1}{2} \rfloor$. By a similar argument, we have $\alpha^*(\mathcal{P}_n^k) = \alpha(\mathcal{P}_n^k) = (k-1)n - \lfloor \frac{n+1}{2} \rfloor + 1$. Also, one can easily see that $\alpha^*(\hat{\mathcal{P}}_n^k) = \alpha(\hat{\mathcal{P}}_n^k) = n + k - 2 - \lfloor \frac{n-1}{k} \rfloor$ and for $n = kr$, $\alpha^*(\hat{\mathcal{C}}_n^k) = \alpha(\hat{\mathcal{C}}_n^k) = n - r$. Thus, for every k, n, r we have $\{\mathcal{C}_n^k, \mathcal{P}_n^k, \hat{\mathcal{C}}_{kr}^k, \hat{\mathcal{P}}_n^k\} \subseteq \mathcal{G}_k$.

A k -uniform hypergraph *corresponding to a given graph* G , $\mathcal{H}_k(G)$, is a hypergraph on $(k-2)|E(G)| + |V(G)|$ vertices and $|E(G)|$ edges where each of its edges is obtained by adding $k-2$ new vertices to an edge of G . For example $\mathcal{P}_n^k = \mathcal{H}_k(P_n)$ and $\mathcal{S}_n^k = \mathcal{H}_k(K_{1,n})$. To see another family of good hypergraphs consider a tree T with the property that the vertices of degree at least 3 are independent. Since $\mathcal{P}_n^k \in \mathcal{G}_k$ one can easily see that $\mathcal{H}_k(T) \in \mathcal{G}_k$. Using Theorem 2.1, we can show that for a bipartite graph G , $\mathcal{H}_k(G) \in \mathcal{G}_k$ if and only if $G \in \mathcal{G}_2$ or equivalently G has a matching saturating one of its color classes. As an example, for a regular bipartite graph G we have $\mathcal{H}_k(G) \in \mathcal{G}_k$.

A k -uniform hypergraph \mathcal{H} is *k -partite* if its vertices can be partitioned into k classes such that each edge intersects any class in exactly one vertex. A k -uniform k -partite hypergraph is called *complete* if it contains all possible edges. A complete k -uniform k -partite hypergraph with part sizes l_1, l_2, \dots, l_k is denoted by $\mathcal{K}(l_1, l_2, \dots, l_k)$. For a complete k -uniform k -partite hypergraph, the vertices outside the smallest part is the maximum independent set and also the strong independent set and so for $l_1 \geq l_2 \geq \dots \geq l_k$,

$$\alpha^*(\mathcal{K}(l_1, l_2, \dots, l_k)) = \alpha(\mathcal{K}(l_1, l_2, \dots, l_k)) = \sum_{i=1}^{k-1} l_i.$$

Hence $\mathcal{K}(l_1, l_2, \dots, l_k) \in \mathcal{G}_k$.

The *Kneser hypergraph* $\mathcal{KH}(n, r, k)$ is a k -uniform hypergraph whose vertices are the r -subsets of a given n -set X and each of its edges contains k mutually disjoint vertices. For $k = 2$, this notion yields the usual definition of Kneser graphs. For $n = rk$ the edges of $\mathcal{KH}(n, r, k)$ containing those r -subsets of X that do not contain a given element of X is a strong independence set of $\mathcal{KH}(n, r, k)$ and so since $\mathcal{KH}(rk, r, k)$ is regular by Corollary 2.2, $\mathcal{KH}(rk, r, k)$ is good. In this case

$$\alpha(\mathcal{KH}(rk, r, k)) = \alpha^*(\mathcal{KH}(rk, r, k)) = \binom{rk-1}{r}.$$

Theorem 2.5 *Let $\mathcal{G} \in \mathcal{F}_k$ and \mathcal{H} be a k -uniform hypergraph with a vertex so that the hypergraph induced by the edges containing this vertex is a star. Then for sufficiently large n ,*

$$R(\mathcal{G}, n\mathcal{H}) \leq |V(\mathcal{G})| + n|V(\mathcal{H})| - \alpha^*(\mathcal{G}) - 1.$$

Proof. Assume that \mathcal{H} has a vertex v so that all edges containing v makes a star with δ edges. Set $l = |V(\mathcal{H})|, l' = |V(\mathcal{G})|, m = (k-1)(R(\mathcal{G}, \mathcal{H}) - 1)l + \alpha^*(\mathcal{G})$ and let

$$n \geq \left(R(\mathcal{G}, \mathcal{K}_m^k) - l' + \alpha^*(\mathcal{G}) + 1 \right) / l.$$

Assume that $p = l' + nl - \alpha^*(\mathcal{G}) - 1$ and consider a 2-edge colored \mathcal{K}_p^k that contains no red copy of \mathcal{G} . Set $V = V(\mathcal{K}_p^k)$. We will show that \mathcal{K}_p^k must contain n disjoint blue copies of \mathcal{H} . First, we observe that $p \geq R(\mathcal{G}, \mathcal{K}_m^k)$ and so we have a blue copy of \mathcal{K}_m^k on a set of vertices B_1 . Find as many disjoint blue copies of \mathcal{H} as possible in the induced hypergraph on $V \setminus B_1$, denoting the vertices of these copies by T_1 and $V \setminus (B_1 \cup T_1)$ by E_1 . Clearly, $|E_1| \leq R(\mathcal{G}, \mathcal{H}) - 1$, since the induced hypergraph on E_1 does not contain a blue \mathcal{H} . If there is a vertex $x \in E_1$ such that the degree of x in $\langle B_1 \cup \{x\} \rangle_{\text{blue}}$ is at least δ , then this vertex and some $l - 1$ vertices of B_1 span a blue \mathcal{H} . Transfer these l vertices to T_1 , and continue this process as long as possible. This yields the three sets E_2, B_2 and T_2 such that there is no vertex $x \in E_2$ with degree at least δ in $\langle B_2 \cup \{x\} \rangle_{\text{blue}}$. Let $t = |E_1 \setminus E_2|$. Clearly $|E_2| \leq R(\mathcal{G}, \mathcal{H}) - t - 1$, $|B_2| = m - t(l - 1)$ and the vertices of T_2 can be partitioned into the disjoint blue copies of \mathcal{H} . Now, a blue edge $e \in E(\mathcal{K}_p^k)$ is called *bad* if $|e \cap B_2| = k - 1$ and $|e \cap E_2| = 1$. For a vertex $v \in E_2$, let W_v be the set of all bad edges containing v and let S_v be the largest star with center v and edges in W_v . Clearly by the condition on E_2 , we have $|V(S_v)| \leq (\delta - 1)(k - 1) + 1$. Now transfer $V(S_v) \setminus \{v\}$ into T_2 and for each $u \in E_2$, continue this process as long as possible. This yields the sets E_3, B_3 and T_3 . Clearly, every $\lceil \frac{|T_3|}{l} \rceil l$ vertices of T_3 can still be partitioned into disjoint blue copies of \mathcal{H} and for any $(k - 1)$ -set S in B_3 and for each vertex $v \in E_3$, the color of the k -set $S \cup \{v\}$ is red. On the other hand,

$$|B_3| \geq |B_2| - |E_2|(k - 1)(\delta - 1) \geq m - t(l - 1) - (R(\mathcal{G}, \mathcal{H}) - t - 1)(k - 1)(\delta - 1) \geq \alpha^*(\mathcal{G}).$$

Therefore, $|E_3| < l' - \alpha^*(\mathcal{G})$ and so $|B_3 \cup T_3| \geq nl$. But then it is clear that the induced hypergraph on $B_3 \cup T_3$ contains n disjoint blue copies of \mathcal{H} . This observation completes the proof. ■

Before giving some applications of Theorem 2.5, we need the following lemma.

Lemma 2.6 *For every $k \geq 2$ and k -uniform hypergraphs \mathcal{G} and \mathcal{H} ,*

$$R(\mathcal{G}, \mathcal{H}) \geq \max\{|V(\mathcal{G})| + |V(\mathcal{H})| - \min\{\alpha(\mathcal{G}), \alpha(\mathcal{H})\} - 1, |V(\mathcal{G})| + \nu(\mathcal{H}) - 1, |V(\mathcal{H})| + \nu(\mathcal{G}) - 1\}.$$

Proof. First we exhibit a 2-coloring, say red and blue, of the edges of the complete k -uniform hypergraph on $|V(\mathcal{G})| + |V(\mathcal{H})| - \min\{\alpha(\mathcal{G}), \alpha(\mathcal{H})\} - 2$ vertices such that this coloring does not contain a red copy of \mathcal{G} and a blue copy of \mathcal{H} . For this purpose, partition the vertex set into two parts A and B , such that $|A| = |V(\mathcal{G})| - \alpha(\mathcal{G}) - 1$ and $|B| = |V(\mathcal{H})| - 1$. We color all edges that contain a vertex of A red, and the rest blue. Now, this coloring can not contain a blue copy of \mathcal{H} , since any such copy must have all vertices in B and $|B| = |V(\mathcal{H})| - 1$. Every red

copy of \mathcal{G} would have to use $\alpha(\mathcal{G}) + 1$ vertices of B , which is impossible since they would all be independent in the red hypergraph. Thus

$$R(\mathcal{G}, \mathcal{H}) \geq |V(\mathcal{G})| + |V(\mathcal{H})| - \alpha(\mathcal{G}) - 1.$$

By symmetry

$$R(\mathcal{G}, \mathcal{H}) \geq |V(\mathcal{G})| + |V(\mathcal{H})| - \alpha(\mathcal{H}) - 1.$$

Combining the two inequalities yields the desired result.

Now partition the vertex set of the complete k -uniform hypergraph on $|V(\mathcal{G})| + \nu(\mathcal{H}) - 2$ vertices into two parts A and B , such that $|A| = \nu(\mathcal{H}) - 1$ and $|B| = |V(\mathcal{G})| - 1$. We color all edges that contain a vertex of A blue, and the rest red. Now, this coloring can not contain a red copy of \mathcal{G} , since any such copy must have all vertices in B and $|B| = |V(\mathcal{G})| - 1$. Also the matching number of the blue hypergraph is at most $\nu(\mathcal{H}) - 1$ and so it can not contain a copy of \mathcal{H} . Hence $R(\mathcal{G}, \mathcal{H}) \geq |V(\mathcal{G})| + \nu(\mathcal{H}) - 1$ and by symmetry again $R(\mathcal{G}, \mathcal{H}) \geq |V(\mathcal{H})| + \nu(\mathcal{G}) - 1$. \blacksquare

The following result is an immediate consequence of Theorem 2.5 and Lemma 3.6.

Theorem 2.7 *Assume that $\mathcal{G} \in \mathcal{G}_k$ and \mathcal{H} is a k -uniform hypergraph with a vertex so that the hypergraph induced by the edges containing this vertex is a star. Then for sufficiently large n ,*

$$R(\mathcal{G}, n\mathcal{H}) = |V(\mathcal{G})| + n|V(\mathcal{H})| - \alpha(\mathcal{G}) - 1.$$

Corollary 2.8 *Assume that $\mathcal{G} \in \mathcal{G}_k$ and \mathcal{H} is a k -uniform linear hypergraph. Then for sufficiently large n ,*

$$R(\mathcal{G}, n\mathcal{H}) = |V(\mathcal{G})| + n|V(\mathcal{H})| - \alpha(\mathcal{G}) - 1.$$

Note that for $\mathcal{G} \in \mathcal{F}_k$, since $\alpha^*(\mathcal{G}) \leq \alpha(\mathcal{G}) \leq |V(\mathcal{G})| - \nu(\mathcal{G})$, the condition $\alpha^*(\mathcal{G}) = |V(\mathcal{G})| - \nu(\mathcal{G})$ implies $\mathcal{G} \in \mathcal{G}_k$ and so we do not have any new result if we consider this condition instead of $\mathcal{G} \in \mathcal{G}_k$ in Theorem 2.7. Using Remark 2.4 and Theorem 2.7, we have the following results.

Theorem 2.9 *Assume that either m or n is sufficiently large, $k \geq 3$ and $\mathcal{G}, \mathcal{H} \in \mathcal{G}_2$. Then*

$$\begin{aligned} R(m\mathcal{C}_r^k, n\mathcal{C}_s^k) &= (k-1)rm + (k-1)sn - \min\{m\alpha(\mathcal{C}_r^k), n\alpha(\mathcal{C}_s^k)\} - 1, \\ R(m\mathcal{P}_r^k, n\mathcal{C}_s^k) &= ((k-1)r+1)m + (k-1)sn - \min\{m\alpha(\mathcal{P}_r^k), n\alpha(\mathcal{C}_s^k)\} - 1, \\ R(m\mathcal{P}_r^k, n\mathcal{P}_s^k) &= ((k-1)r+1)m + ((k-1)s+1)n - \min\{m\alpha(\mathcal{P}_r^k), n\alpha(\mathcal{P}_s^k)\} - 1, \\ R(m\hat{\mathcal{P}}_r^k, n\hat{\mathcal{P}}_s^k) &= (r+k-1)m + (s+k-1)n - \min\{m\alpha(\hat{\mathcal{P}}_r^k), n\alpha(\hat{\mathcal{P}}_s^k)\} - 1, \\ R(m\mathcal{S}_r^k, n\mathcal{S}_s^k) &= (k-1)(mr+ns) + m + n - \min\{mr(k-1), ns(k-1)\} - 1, \\ R(m\mathcal{G}, n\mathcal{H}) &= m|V(\mathcal{G})| + n|V(\mathcal{H})| - \min\{m\alpha(\mathcal{G}), n\alpha(\mathcal{H})\} - 1. \end{aligned}$$

Theorem 2.10 *Assume that \mathcal{G} is a bipartite graph with a matching saturating one of its color classes and \mathcal{H} is an arbitrary graph. Then for sufficiently large n ,*

$$R(\mathcal{G}, n\mathcal{H}) = n|V(\mathcal{H})| + \nu(\mathcal{G}) - 1.$$

3 Multiple copies of loose paths and cycles

In this section, we provide the exact values of $R(m\mathcal{G}, n\mathcal{H})$ for every $m \geq n$ and particular hypergraphs \mathcal{G} and \mathcal{H} with $\alpha(\mathcal{G}) \geq \alpha(\mathcal{H})$, for example, k -uniform loose triangles, loose quadrangles, stars with maximum degree 3 and 3-uniform loose paths and cycles. Before that, we need the following.

Lemma 3.1 *For every $k \geq 2$ and k -uniform hypergraphs \mathcal{G} , \mathcal{H} and \mathcal{F} ,*

$$R(\mathcal{G}, \mathcal{H} \cup \mathcal{F}) \leq \max\{R(\mathcal{G}, \mathcal{F}) + |V(\mathcal{H})|, R(\mathcal{G}, \mathcal{H})\},$$

$$R(m\mathcal{G}, n\mathcal{H}) \leq R(\mathcal{G}, \mathcal{H}) + (m-1)|V(\mathcal{G})| + (n-1)|V(\mathcal{H})|.$$

Proof. Let $r = \max\{R(\mathcal{G}, \mathcal{F}) + |V(\mathcal{H})|, R(\mathcal{G}, \mathcal{H})\}$ and the edges of $K = \mathcal{K}_r^k$ be 2-colored red and blue. If there is no red \mathcal{G} , then there must certainly be a blue \mathcal{H} . Remove the vertices of this blue copy of \mathcal{H} from K . Among the remaining vertices there must be a blue \mathcal{F} or a red \mathcal{G} . Hence K contains either a red \mathcal{G} or a blue $\mathcal{H} \cup \mathcal{F}$, and the first inequality follows. The second inequality follows by applying the first inequality. ■

Theorem 3.2 *Assume that $k \geq 2$, $m \geq n \geq 2$ and \mathcal{G} and \mathcal{H} are arbitrary k -uniform hypergraphs. Then*

$$R(n\mathcal{G}, m\mathcal{H}) \leq R((n-1)\mathcal{G}, (m-1)\mathcal{H}) + R(\mathcal{G}, \mathcal{H}) + 1.$$

In particular,

$$R(n\mathcal{G}, m\mathcal{H}) \leq R(\mathcal{G}, (m-n+1)\mathcal{H}) + (n-1)(R(\mathcal{G}, \mathcal{H}) + 1).$$

Proof. Set $t = R((n-1)\mathcal{G}, (m-1)\mathcal{H}) + R(\mathcal{G}, \mathcal{H}) + 1$ and let $K = \mathcal{K}_t^k$ be 2-edge colored red and blue. We find either a red $n\mathcal{G}$ or a blue $m\mathcal{H}$. We have $t \geq R(\mathcal{G}, \mathcal{H})$ and thus we may assume that K contains a red copy of \mathcal{G} (we have the same proof if K contains a blue copy of \mathcal{H}). Discard this copy. Since the number of remaining vertices is greater than $R((n-1)\mathcal{G}, (m-1)\mathcal{H})$, there is either a red copy of $(n-1)\mathcal{G}$ or a blue copy of $(m-1)\mathcal{H}$. In the first case, we have a red $n\mathcal{G}$ and so we are done. Thus we may assume that K contains a blue $(m-1)\mathcal{H}$ and therefore we have a red copy of \mathcal{G} and a blue copy of \mathcal{H} . Among red-blue copies of \mathcal{G} and \mathcal{H} choose red-blue copies with maximum intersection. Let \mathcal{G}' and \mathcal{H}' be such copies. We must have $p = |V(\mathcal{G}' \cup \mathcal{H}')| \leq R(\mathcal{G}, \mathcal{H}) + 1$. Indeed, let $p \geq R(\mathcal{G}, \mathcal{H}) + 2$. Clearly $p \geq \max\{|V(\mathcal{G}')|, |V(\mathcal{H}')|\} + 2$ and hence $V(\mathcal{G}') \setminus V(\mathcal{H}')$ and $V(\mathcal{H}') \setminus V(\mathcal{G}')$ are non-empty. Choose $v_1 \in V(\mathcal{G}') \setminus V(\mathcal{H}')$ and $v_2 \in V(\mathcal{H}') \setminus V(\mathcal{G}')$. Set $U = (V(\mathcal{G}') \cup V(\mathcal{H}')) \setminus \{v_1, v_2\}$. Since $|U| = p - 2 \geq R(\mathcal{G}, \mathcal{H})$, we have either a red \mathcal{G} or a blue \mathcal{H} , say \mathcal{F} . If \mathcal{F} is red, then $|\mathcal{F} \cap \mathcal{H}'| > |\mathcal{G}' \cap \mathcal{H}'|$. If \mathcal{F} is blue, then $|\mathcal{F} \cap \mathcal{G}'| > |\mathcal{G}' \cap \mathcal{H}'|$. Both cases, contradict the choice of \mathcal{G}' and \mathcal{H}' . Therefore $|V(\mathcal{G}' \cap \mathcal{H}')| \leq R(\mathcal{G}, \mathcal{H}) + 1$. Remove the vertices of $\mathcal{G}' \cup \mathcal{H}'$ from K . The hypergraph on the remaining vertices contains either a red $(n-1)\mathcal{G}$ or a blue $(m-1)\mathcal{H}$, say T , to which we add the appropriately colored copy of \mathcal{G} and \mathcal{H} to obtain a red $n\mathcal{G}$ or a blue $m\mathcal{H}$. This observation completes the proof of the first inequality. The second inequality follows from repeated application of the first inequality, which completes the proof. ■

As an easy, but useful application of Theorem 3.2, we have the following corollary.

Corollary 3.3 Assume that $k \geq 2$, $n \geq 1$ and \mathcal{G} and \mathcal{H} are k -uniform hypergraphs. Then

$$R(n\mathcal{G}, n\mathcal{H}) \leq (R(\mathcal{G}, \mathcal{H}) + 1)n - 1.$$

Using Lemma 3.6 and Corollary 3.3, we get the following theorem.

Theorem 3.4 Assume that $k \geq 2$ and $m \geq n \geq 1$ are positive integers and \mathcal{G} and \mathcal{H} are k -uniform hypergraphs with $\alpha(\mathcal{G}) \geq \alpha(\mathcal{H})$ and $R(\mathcal{G}, \mathcal{H}) = |V(\mathcal{G})| + |V(\mathcal{H})| - \alpha(\mathcal{H}) - 1$. Then

$$R(m\mathcal{G}, n\mathcal{H}) = m|V(\mathcal{G})| + n|V(\mathcal{H})| - n\alpha(\mathcal{H}) - 1.$$

Proof. Let $t = m|V(\mathcal{G})| + n|V(\mathcal{H})| - n\alpha(\mathcal{H}) - 1$ and $K = \mathcal{K}_t^k$ be 2-edge colored red and blue. We use induction on m to prove that either $m\mathcal{G} \subseteq K_{red}$ or $n\mathcal{H} \subseteq K_{blue}$. Clearly for $m = 1$, the result is true and so we may assume that $m \geq 2$. For $m = n$, the result follows from Corollary 3.3 and so we may assume that $m - 1 \geq n$. Since by the induction hypothesis

$$R((m-1)\mathcal{G}, n\mathcal{H}) \leq (m-1)|V(\mathcal{G})| + n|V(\mathcal{H})| - n\alpha(\mathcal{H}) - 1 < t,$$

we may assume that we have $(m-1)\mathcal{G} \subseteq K_{red}$, otherwise we can find n disjoint blue copies of \mathcal{H} . Now, remove the vertices of a red \mathcal{G} from K and use the induction hypothesis to the coloring on the remaining $(m-1)|V(\mathcal{G})| + n|V(\mathcal{H})| - n\alpha(\mathcal{H}) - 1$ vertices to find either a $(m-1)\mathcal{G} \subseteq K_{red}$ or a $n\mathcal{H} \subseteq K_{blue}$. If $n\mathcal{H} \subseteq K_{blue}$ we are done, otherwise $(m-1)\mathcal{G} \subseteq K_{red}$, adding the deleted red colored \mathcal{G} to the red $(m-1)\mathcal{G}$, we obtain m disjoint red copies of \mathcal{G} , which shows that $R(m\mathcal{G}, n\mathcal{H}) \leq t$.

For the lower bound, employ Lemma 3.6, where \mathcal{G} is replaced by $m\mathcal{G}$ and \mathcal{H} is replaced by $n\mathcal{H}$, which completes the proof. ■

In the rest of this section, we use Theorem 3.4 to give some corollaries. Before that, we complete the determining of the Ramsey numbers of loose paths and cycles with at most 4 edges.

Lemma 3.5 For every $k \geq 3$, $R(\mathcal{C}_3^k, \mathcal{C}_4^k) = 4k - 3$ and $R(\mathcal{S}_3^k, \mathcal{S}_3^k) = 3k - 2$.

Proof. Using Lemma 3.6, $R(\mathcal{C}_3^k, \mathcal{C}_4^k) \geq 4k - 3$ and $R(\mathcal{S}_3^k, \mathcal{S}_3^k) \geq 3k - 2$. To prove that $R(\mathcal{C}_3^k, \mathcal{C}_4^k) \leq 4k - 3$, suppose that the edges of $K = \mathcal{K}_{4k-3}^k$ are arbitrary colored red and blue. We prove that K contains a red copy of \mathcal{C}_3^k or a blue copy of \mathcal{C}_4^k . Since $R(\mathcal{C}_4^k, \mathcal{C}_4^k) = 4k - 3$, we may assume that K contains a red copy of \mathcal{C}_4^k . Let $e_i = \{v_1, v_2, \dots, v_k\} + i(k-1) \bmod 4$, $0 \leq i \leq 3$, be the edges of $\mathcal{C}_4^k \subseteq K_{red}$ and v be the remaining vertex which is not covered by this copy of \mathcal{C}_4^k . Set $e'_0 = \{v_1, v_{2k-1}, \dots, v_{3k-3}\}$, $e'_1 = \{v_{3k-3}, v_{4k-4}, v_2, \dots, v_{k-2}, v\}$, $e'_2 = \{v_{2k-2}, v_{3k-2}, v_{3k}, v_{3k+1}, \dots, v_{4k-4}, v_k\}$ and $e'_3 = \{v_{3k-1}, v_{k-1}, v_{k+2}, \dots, v_{2k-1}\}$. If one of e'_i is red, we have a red copy of \mathcal{C}_3^k , otherwise e'_0, e'_1, e'_2, e'_3 form a blue copy of \mathcal{C}_4^k which shows that $R(\mathcal{C}_3^k, \mathcal{C}_4^k) \leq 4k - 3$. To see $R(\mathcal{S}_3^k, \mathcal{S}_3^k) \leq 3k - 2$, let the edges of $K = \mathcal{K}_{3k-2}^k$ be arbitrary colored red and blue. By Theorem 1.1, we have a monochromatic, say red, copy of \mathcal{C}_3^k . Assume that $e_1 = \{v, v_1, \dots, v_{k-2}, u\}$, $e_2 = \{u, u_1, \dots, u_{k-2}, w\}$ and $e_3 = \{w, w_1, \dots, w_{k-2}, v\}$ are the edges of this copy of \mathcal{C}_3^k and $T = \{t\}$ be the remaining vertex of K . If one of the edges $e'_1 = \{t, w, v_1, \dots, v_{k-2}\}$, $e'_2 = \{t, u, w_1, \dots, w_{k-2}\}$ or $e'_3 = \{t, v, u_1, \dots, u_{k-2}\}$ is red, then we have a red copy of $\mathcal{S}_3^k \subseteq K_{red}$, otherwise $e'_1 e'_2 e'_3$ form a $\mathcal{S}_3^k \subseteq K_{blue}$. This observation completes the proof. ■

Lemma 3.6 ([7]) *Let $n \geq m \geq 3$, $k \geq 3$ and $r = (k-1)n + \lfloor \frac{m+1}{2} \rfloor$. Then*
(i) $R(\mathcal{C}_n^k, \mathcal{C}_m^k) \geq r-1$ *and also r is a lower bound for both $R(\mathcal{P}_n^k, \mathcal{P}_m^k)$ and $R(\mathcal{P}_n^k, \mathcal{C}_m^k)$.*
(ii) *Assume that \mathcal{K}_r^k is 2-edge colored red and blue. If $\mathcal{C}_n^k \subseteq \mathcal{F}_{red}$, then either $\mathcal{P}_n^k \subseteq \mathcal{F}_{red}$ or $\mathcal{P}_m^k \subseteq \mathcal{F}_{blue}$. Also, if $\mathcal{C}_n^k \subseteq \mathcal{F}_{red}$, then either $\mathcal{P}_n^k \subseteq \mathcal{F}_{red}$ or $\mathcal{C}_m^k \subseteq \mathcal{F}_{blue}$.*

The following theorem is a direct consequence of Lemmas 3.5 and 3.6 and Theorem 1.1.

Lemma 3.7 *For every $k \geq 3$, we have*

$$R(\mathcal{P}_3^k, \mathcal{P}_4^k) = R(\mathcal{C}_3^k, \mathcal{P}_4^k) = 4k-2,$$

$$R(\mathcal{P}_3^k, \mathcal{C}_4^k) = 4k-3, \quad R(\mathcal{C}_3^k, \mathcal{P}_3^k) = 3k-1.$$

Now, using the known result $R(P_r, P_s) = r + \lfloor \frac{s}{2} \rfloor - 1$ for $r \geq s$ due to Gerencsér and Gyárfás in [6] and Theorems 1.1, 1.2 and 3.4 and Lemmas 3.5 and 3.7, we have the following theorems.

Theorem 3.8 *If $m \geq n \geq 1$ and $k \geq 3$, then*

- (i) $R(m\mathcal{C}_3^k, n\mathcal{C}_3^k) = m(3k-3) + 2n-1$ *and* $R(m\mathcal{C}_4^k, n\mathcal{C}_3^k) = R(m\mathcal{C}_4^k, n\mathcal{C}_4^k) = m(4k-4) + 2n-1$,
- (ii) $R(m\mathcal{P}_3^k, n\mathcal{C}_3^k) = m(3k-2) + 2n-1$ *and* $R(m\mathcal{P}_4^k, n\mathcal{C}_3^k) = R(m\mathcal{P}_4^k, n\mathcal{C}_4^k) = m(4k-3) + 2n-1$,
- (iii) $R(m\mathcal{P}_3^k, n\mathcal{P}_3^k) = m(3k-2) + 2n-1$ *and* $R(m\mathcal{P}_4^k, n\mathcal{P}_3^k) = R(m\mathcal{P}_4^k, n\mathcal{P}_4^k) = m(4k-3) + 2n-1$,
- (iv) $R(m\mathcal{S}_3^k, n\mathcal{S}_3^k) = m(3k-2) + n-1$,
- (v) *For every k -uniform hypergraph \mathcal{H} , $R(m\mathcal{H}, n\mathcal{K}_k^k) = m|V(\mathcal{H})| + n-1$. In particular $R(m\mathcal{K}_k^k, n\mathcal{K}_k^k) = mk + n-1$.*

Theorem 3.9 *For every $m \geq n \geq 1$ and $r \geq s \geq 1$ we have the following.*

- (i) $R(m\mathcal{P}_r^3, n\mathcal{P}_s^3) = R(m\mathcal{P}_r^3, n\mathcal{C}_s^3) = (2r+1)m + \lfloor \frac{s+1}{2} \rfloor n - 1$,
- (ii) $R(m\mathcal{C}_r^3, n\mathcal{C}_s^3) = 2rm + \lfloor \frac{s+1}{2} \rfloor n - 1$,
- (iii) $R(m\mathcal{C}_r^3, n\mathcal{P}_s^3) = 2rm + \lfloor \frac{s+1}{2} \rfloor n - 1$ *if $r > s$,*
- (iv) $R(m\mathcal{P}_r, n\mathcal{P}_s) = rm + \lfloor \frac{s}{2} \rfloor n - 1$.

4 Concluding remarks

By Lemma 3.6, for every $k \geq 3$, $m \geq n$ and $r \geq s \geq 3$,

$$R(m\mathcal{C}_r^k, n\mathcal{C}_s^k) \geq (k-1)rm + \left\lfloor \frac{s+1}{2} \right\rfloor n - 1. \quad (2)$$

By Theorem 2.9, we have equality in (2) if m is sufficiently large. It would be interesting to decide whether this natural lower bound is always the exact value of the Ramsey number. The case $k=3$ follows from Theorem 3.9. Based on these observations and with the same discussions for the Ramsey numbers of multiple copies of hypergraphs involving loose paths, loose cycles, tight paths and tight cycles we pose the following conjecture.

Conjecture 1 For every $k \geq 3$, $m \geq n$ and $r \geq s \geq 3$,

$$R(m\mathcal{P}_r^k, n\mathcal{P}_s^k) = R(m\mathcal{P}_r^k, n\mathcal{C}_s^k) = ((k-1)r+1)m + \left\lfloor \frac{s+1}{2} \right\rfloor n - 1,$$

$$R(m\hat{\mathcal{P}}_r^k, n\hat{\mathcal{P}}_s^k) = (r+k-1)m + (1 + \left\lfloor \frac{s-1}{k} \right\rfloor)n - 1,$$

$$R(m\mathcal{C}_r^k, n\mathcal{C}_s^k) = (k-1)rm + \left\lfloor \frac{s+1}{2} \right\rfloor n - 1,$$

and if $r > s$

$$R(m\mathcal{C}_r^k, n\mathcal{P}_s^k) = (k-1)rm + \left\lfloor \frac{s+1}{2} \right\rfloor n - 1.$$

Using Theorem 2.7, for a natural number m , a k -uniform hypergraph \mathcal{H} , a given $\mathcal{G} \in \mathcal{G}_k$ and sufficiently large n we have

$$R(m\mathcal{G}, n\mathcal{H}) = m|V(\mathcal{G})| + n|V(\mathcal{H})| - m\alpha(\mathcal{G}) - 1.$$

Based on this equality, we pose the following conjecture for the Ramsey number of k -uniform hypergraphs corresponding to trees.

Conjecture 2 For $k \geq 3$, assume that $\mathcal{G} = \mathcal{H}_k(T)$ and $\mathcal{H} = \mathcal{H}_k(T')$ where T and T' are trees in \mathcal{G}_2 . If $\alpha(\mathcal{G}) \leq \alpha(\mathcal{H})$ and $m \leq n$, then

$$R(m\mathcal{G}, n\mathcal{H}) = m|V(\mathcal{G})| + n|V(\mathcal{H})| - m\alpha(\mathcal{G}) - 1.$$

Using Theorems 3.8 and 3.9, Conjecture 2 is true for an arbitrary k when T and T' are either two paths with at most 4 edges or stars with 3 edges and also for $k = 3$ when T and T' are two arbitrary paths. Clearly by Theorem 3.4, if this conjecture holds for the case $m = n = 1$, then it holds for every $m \geq n \geq 1$.

References

- [1] S.A. Burr, P. Erdős and J.H. Spencer, Ramsey theorems for multiple copies of graphs, *Trans. Amer. Math. Soc.* **209** (1975), 87-99.
- [2] S.A. Burr, On the Ramsey numbers $r(G, nH)$ and $r(nG, nH)$ when n is large, *Discrete Math.* **65** (1987), 215-229.
- [3] S.A. Burr, On Ramsey numbers for large disjoint unions of graphs, *Discrete Math.* **70** (1988), 277-293.
- [4] R.J. Faudree, S.L. Lawrence, T.D. Parsons and R.H. Schelp, Path-cycle Ramsey numbers, *Discrete Math.* **10** (1974), 269-277.
- [5] R.J. Faudree and R.H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Math.* **8** (1974), 313-329.
- [6] L. Gerencsér and A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **10** (1967), 167-170.

- [7] A. Gyárfás and G. Raeisi, The Ramsey number of loose triangles and quadrangles in hypergraphs, *Electron. J. Combin.* **19** (2012), no. 2, #R30.
- [8] A. Gyárfás, G. Sárközy and E. Szemerédi, The Ramsey number of diamond-matchings and loose cycles in hypergraphs, *Electron. J. Combin.* **15** (2008), no. 1, #R126.
- [9] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, M. Simonovits and J. Skokan, The Ramsey number for hypergraph cycles I, *J. Combin. Theory, Ser. A*, **113** (2006), 67-83.
- [10] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, and J. Skokan, The Ramsey number for 3-uniform tight hypergraph cycles, *Combin. Prob. Comput.*, **18** (2009), 165-203.
- [11] M. Maherani, G.R. Omid, G. Raeisi and M. Shahsiah, The Ramsey number of loose paths in 3-uniform hypergraphs, *Electron. J. Combin.* **20** (2013), no. 1, #P12.
- [12] G.R. Omid and M. Shahsiah, Ramsey numbers of 3-uniform loose paths and loose cycles, *Submitted*.
- [13] S.P. Radziszowski, Ramsey numbers involving cycles, in *Ramsey Theory, Yesterday, Today and Tomorrow*, A. Soifer ed., Progress in Mathematics **285**.
- [14] F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc. 2nd Ser.* **30** (1930), 264-286.
- [15] V. Rosta, On a Ramsey type problem of J.A. Bondy and P. Erdős, I & II, *J. Combin. Theory, Ser. B*, **15** (1973), 94-120.